Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 6

Exercise 1 *(4 Points)*
Let $(b_n)_{n\geq 1} \subset \mathbb{R}$ and $(\sigma_n)_{n\geq 1} \subset (0,\infty)$. Define $\mu_n(dx) = p_n(x)dx$ with

$$p_n(x) = \sqrt{\frac{1}{2\pi\sigma_n^2}} e^{-\frac{(x-b_n)^2}{2\sigma_n^2}}, \quad x \in \mathbb{R}, \quad n \geq 1.$$

Suppose that there exist $b \in \mathbb{R}$ and $\sigma \in [0,\infty)$ such that $b_n \longrightarrow b$ and $\sigma_n \longrightarrow \sigma$. Prove that there exists a probability measure μ on $\mathbb{R}$ (which depends on b and σ) such that $\mu_n \longrightarrow \mu$ weakly. Under which condition on b and σ does μ have a density with respect to the Lebesgue measure?

Exercise 2 *(4 Points)*
Let $E = [0,1]$ and let $\mu_n(dx)$ be the uniform distribution on $\{\frac{k}{n} \mid k = 1,\ldots,n\}$, i.e.

$$\mu_n(dx) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\frac{k}{n}}(dx).$$

Prove that μ_n converges weakly and find its limit.

Exercise 3 *(4 Points)*
Let (E,d) be a metric space and $\mu \in \mathcal{P}(E)$ given by

$$\mu = \sum_{k=1}^{\infty} p_k \delta_{x_k},$$

where $(x_k)_{k\in\mathbb{N}} \subset E$ with $x_i \neq x_j$, $i \neq j$, and $p_k \in [0,1]$ with $\sum_{k=1}^{\infty} p_k = 1$. Show that μ is regular. *Hint:* Use that any finite collection of points $\{y_1,\ldots,y_m\} \subset E$ is compact. Moreover, use that the intersection of a closed and a compact set is compact.

Exercise 4 *(4 Points, talk)*
Prepare a short talk on the main definitions of independence for *sets, σ-algebras and random variables.*